INCREASING AND DECREASING SEQUENCES OF LENGTH TWO IN 01-FILLINGS OF MOON POLYOMINOES

Anisse Kasraoui

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ANISSE KASRAOUI

Université de Lyon ; Université Lyon 1
Institut Camille Jordan CNRS UMR 5208
43, boulevard du 11 novembre 1918
F-69622, Villeurbanne Cedex
anisse@math.univ-lyon1.fr

Abstract. We put recent results on the symmetry of the joint distribution of the numbers of crossings and nestings of two edges over matchings and set partitions in the larger context of the enumeration of increasing and decreasing sequences of length 2 in fillings of moon polyominoes.

1. Introduction

The main purpose of this paper is to put recent results of Klazar and Noy [10], Kasraoui and Zeng [9], and Chen, Wu and Yan [2], on the enumeration of 2-crossings and 2-nestings in matchings, set partitions and linked partitions in the larger context of enumeration of increasing and decreasing chains in fillings of arrangements of cells. Our work is motivated by the recent paper of Krattenthaler [11] in which results of Chen et al. [3] on the symmetry of the crossing number and nesting number in matchings and set partitions have been extended in a such context.

Let G be a *simple graph* (no multiple edges and loops) on $[n] := \{1, 2, \ldots, n\}$. A graph will be represented by its set of edges where the edge $\{i, j\}$ is written (i, j) if $i < j$.

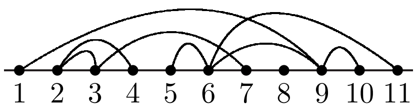


Figure 1. The graph $\{(1, 9), (2, 3), (2, 4), (3, 7), (5, 6), (6, 9), (6, 11), (9, 10)\}$

A sequence $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$ of edges of G is said to be a *k-crossing* if $i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k$ and a *k-nesting* if $i_1 < i_2 < \cdots < i_k < j_k < \cdots < j_2 < j_1$. If we draw the vertices of G in increasing order on a line and draw the arcs above the line (see Figure 1 for an illustration), k-crossings and k-nestings have a nice geometrical meaning. The largest k for which a graph G has a k-crossing (resp., a k-nesting) is denoted $\mathrm{cros}(G)$ (resp., $\mathrm{nest}(G)$) and called [3] the *crossing number* (resp., *nesting number*) of G. The number of k-crossings (resp., k-nestings) of G will be denoted by $\mathrm{cros}_k(G)$ (resp., $\mathrm{nest}_k(G)$). A graph with no k-crossing is called *k-noncrossing* and a graph with no k-nesting is called *k-nonnesting*. As usual, a 2-noncrossing (resp., 2-nonnesting) graph is just said to be *noncrossing* (resp., *nonnesting*). Recently, there has been an increasing interest in studying crossings and nestings in matchings, set partitions, linked partitions and permutations (see e.g. [1, 2, 3, 4, 5, 6, 9, 10, 12]).

A (*set*) *partition* of $[n]$ is a collection of non-empty pairwise disjoint sets, called blocks, whose union is $[n]$. A (*complete*) *matching* of $[n]$ is just a set partition whose each block contains exactly two elements. The set of all set partitions and matchings of $[n]$ will be denoted respectively by $\mathcal{P}_n$ and $\mathcal{M}_n$. Set partitions (and thus matchings) have a natural graphical representation, called *standard representation.* To each set partition π of $[n]$, one associates the graph St_π on $[n]$ whose edge set consists of arcs joining the elements of each block in numerical order. For instance, the standard representation of the set partition $\pi = \{\{1,9,10\},\{2,3,7\},\{4\},\{5,6,11\},\{8\}\}$ is the graph on $\{1,2,\dots,11\}$ $St_\pi = \{(1,9),(9,10),(2,3),(3,7),(5,6),(6,11)\}$ drawn in Figure 2.

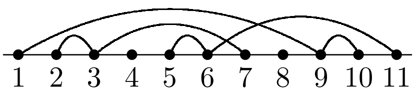


FIGURE 2. Standard representation of $\pi = \{1,9,10\}\{2,3,7\}\{4\}\{5,6,11\}\{8\}$

Throughout this paper, set partitions (and matchings) will be identified with their standard representation. It is well-known that the number of noncrossing matchings of $[2n]$ equals the number of nonnesting matchings of $[2n]$, and that the number of noncrossing partitions of $[n]$ equals the number of nonnesting partitions of $[n]$ (and these are the n-th Catalan number), i.e.

$$|\{M \in \mathcal{M}_{2n} : \mathrm{cros}_2(M) = 0\}| = |\{M \in \mathcal{M}_{2n} : \mathrm{nest}_2(M) = 0\}|, \tag{1.1}$$

$$|\{\pi \in \mathcal{P}_n : \mathrm{cros}_2(\pi) = 0\}| = |\{\pi \in \mathcal{P}_n : \mathrm{nest}_2(\pi) = 0\}|. \tag{1.2}$$

In recent works, two generalizations of the latter identities have been investigated. The first one is an extension of results obtained by De Sainte-Catherine, and Klazar and Noy on the distributions of 2-crossings and 2-nestings on matchings. In her thesis [7], De Sainte-Catherine have shown that the statistics cros_2 and nest_2 are equidistributed over all matchings of $[2n]$, that is for any integer $\ell \geq 0$,

$$|\{M \in \mathcal{M}_{2n} : \mathrm{cros}_2(M) = \ell\}| = |\{M \in \mathcal{M}_{2n} : \mathrm{nest}_2(M) = \ell\}|, \tag{1.3}$$

i.e. in other words,

$$\sum_{M \in \mathcal{M}_{2n}} p^{\mathrm{cros}_2(M)} = \sum_{M \in \mathcal{M}_{2n}} p^{\mathrm{nest}_2(M)}. \tag{1.4}$$

Klazar and Noy [10] have shown that (1.1) is even more true because the distribution of the joint statistic $(\mathrm{cros}_2, \mathrm{nest}_2)$ is symmetric over $\mathcal{M}_{2n}$ that is

$$\sum_{M \in \mathcal{M}_{2n}} p^{\mathrm{cros}_2(M)} q^{\mathrm{nest}_2(M)} = \sum_{M \in \mathcal{M}_{2n}} p^{\mathrm{nest}_2(M)} q^{\mathrm{cros}_2(M)}. \tag{1.5}$$

Equations (1.2) and (1.4)(1.5) motivates Kasraoui and Zeng to pose and solve the following questions: Are the statistics cros_2 and nest_2 equidistributed over all partitions of $[n]$? Is the distribution of the joint statistic $(\mathrm{cros}_2, \mathrm{nest}_2)$ symmetric over all partitions of $[n]$? For S, T two subsets of $[n]$, let $\mathcal{P}_n(S,T)$ be the set of all partitions of $[n]$ whose the set of lefthand (resp., righthand) endpoints of the arcs of π is equal to S (resp., T). For instance, the set partition drawn in Figure 2 belong to $\mathcal{P}_n(S,T)$, with $S=\{1,2,3,5,6,9\}$ and $T=\{3,6,7,9,10,11\}$. Generalizing Klazar and Noy's result (1.5), Kasraoui and Zeng [9] have proved that the distribution of the joint statistic $(\mathrm{cros}_2, \mathrm{nest}_2)$ is symmetric over each $\mathcal{P}_n(S,T)$ (and thus, over $\mathcal{P}_n$ and $\mathcal{M}_n$), that is

$$\sum_{\pi\in\mathcal{P}_n(S,T)} p^{\mathrm{cros}_2(\pi)}q^{\mathrm{nest}_2(\pi)} = \sum_{\pi\in\mathcal{P}_n(S,T)} p^{\mathrm{nest}_2(\pi)}q^{\mathrm{cros}_2(\pi)}. \tag{1.6}$$

Note that recently, Chen, Wu and Yan [2] have generalized the above result (although it is not explicitly stated) by considering linked set partitions (see (3.3)).

The second generalization of (1.4) and (1.5) is due to Chen, Deng, Du, Stanley and Yan [3]. It states, remarkably, that for any $k\geq 2$ the number of k-noncrossing partitions (resp. matchings) of $[n]$ equals the number of k-nonnesting partitions (resp. matchings) of $[n]$. More generally, Chen et al. [3] have proved that the distribution of the joint statistic $(\mathrm{cros}, \mathrm{nest})$ is symmetric over each $\mathcal{P}_n(S,T)$, that is

$$\sum_{\pi\in\mathcal{P}_n(S,T)} p^{\mathrm{cros}(\pi)}q^{\mathrm{nest}(\pi)} = \sum_{\pi\in\mathcal{P}_n(S,T)} p^{\mathrm{nest}(\pi)}q^{\mathrm{cros}(\pi)}. \tag{1.7}$$

In the recent paper [11], Krattenthaler have put Chen et al's result (1.7) in the larger context of the enumeration of increasing and decreasing chains in fillings of ferrers shapes. First recall the correspondence between simple graphs of $[n]$ and 01-fillings of Δ_n, the triangular shape with $n-1$ cells in the bottom row, $n-2$ cells in the row above, etc., and 1 cell in the top-most row. See Figure 3 for an example in which $n=11$ (the filling and labeling of the corners should be ignored at this point. For convenience, we also joined pending edges at the right and at the top of Δ_n). Let G be a simple graph of $[n]$. The correspondence consists in labeling in increasing order columns from left to right by $\{1,2,\ldots,n\}$ and rows from top to bottom by $\{1,2,\ldots,n\}$. Then assign the value 1 to the cell on column labeled i and row labeled j if and only if (i,j) is an edge of G. An illustration is given in Figure 3.

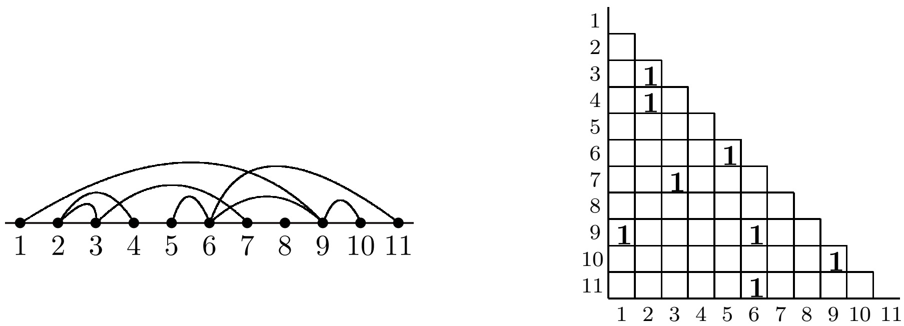


FIGURE 3. A graph and the corresponding 01-filling

It is obvious to see that in this correspondence a k-crossing (resp., k-nesting) corresponds to a SE-chain (resp., NE-chain) of length k such that the smallest rectangle containing the chain is contained in Δ_n, a SE-chain (resp., NE-chain) of length k being a sequence of k 1's in the filling such that any 1 in the sequence is below and to the right (resp., above and to the right) of the preceding 1 in the sequence. Moreover, it is obvious that this correspondence establishes a bijection between set partitions of $[n]$ and $\mathcal{N}(\Delta_n)$, the set of all 01-fillings of Δ_n in which every row and every column contains at most one 1. Therefore, Kasraoui and Zeng's result (1.6) and Chen et al's result (1.7) can be viewed as a property of symmetry of NE-chains and SE-chains over $\mathcal{N}(\Delta_n)$. Given a 01-filling F of Δ_n, denote by $\mathrm{se}(F)$ (resp., $\mathrm{ne}(F)$) the maximal k such that F has a SE-chain (resp., NE-chain) of length k, the smallest rectangle containing the chain being contained in F and by $\mathrm{se}_2(F)$ (resp., $\mathrm{ne}_2(F)$) the number of SE-chains (resp., NE-chains) of length 2 such that the smallest rectangle containing the chain is contained in F. Then the symmetry of the distributions of the joint statistics $(\mathrm{cros}_2, \mathrm{nest}_2)$ and $(\mathrm{cros}, \mathrm{nest})$ over set partitions can be reformulated respectively as follows:

$$\sum_{F\in\mathcal{N}(\Delta_n)} p^{\mathrm{ne}_2(F)} q^{\mathrm{se}_2(F)} = \sum_{F\in\mathcal{N}(\Delta_n)} p^{\mathrm{se}_2(F)} q^{\mathrm{ne}_2(F)} \tag{1.8}$$

$$\sum_{F\in\mathcal{N}(\Delta_n)} p^{\mathrm{ne}(F)} q^{\mathrm{se}(F)} = \sum_{F\in\mathcal{N}(\Delta_n)} p^{\mathrm{se}(F)} q^{\mathrm{ne}(F)}. \tag{1.9}$$

In the recent paper [11], Krattenthaler have shown that (1.9) remains true if we replace Δ_n by any ferrers shapes and he proposed to investigate more general arrangements. This was done successfully by Rubey [13] for moon polyominoes. It is thus natural to ask if (1.8) remains true when we replace Δ_n by any ferrers shape or, more generally by any moon polyomino (We will answer this question by the affirmative): this is the original motivation of our paper.

2. The main results

A *polyomino* is an arrangement of square cells. It is *convex* if along any row of cells and along any column of cells there is no hole. It is *intersection free* if any two rows are comparable, i.e., one row can be embedded in the other by applying a vertical shift. Equivalently, it is *intersection free* if any two columns are comparable, i.e., one row can be embedded in the other by applying an horizontal shift. A *moon polyomino* is a convex and intersection free polyomino. An illustration is given in Figure 4.

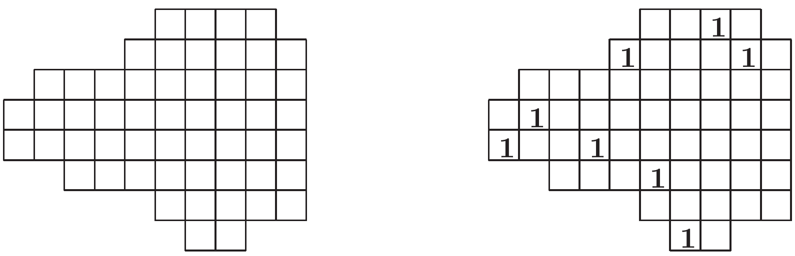


Figure 4. A moon polyomino T and a 01-filling of T.

Let T be a moon polyomino. A 01-*filling* F of T consists of assigning 0 or 1 to each cell. For convenience, we will omit the 0's when we draw the fillings. See Figure 4. The set of all 01-fillings of T will be denoted $\mathcal{N}^{01}(T)$. Recall that a SE-chain (resp., NE-chain) of length k in a 01-filling F of T is a sequence of k 1's in the filling such that any 1 in the sequence is below and to the right (resp., above and to the right) of the preceding 1 in the sequence. A SE-chain (resp., NE-chain) of length 2 such that the smallest rectangle containing the chain is contained in F is said to be a *descent* (resp., an *ascent*). We will denote by $\mathrm{se}_2(F)$ and $\mathrm{ne}_2(F)$ the number of descents and ascents in F. For instance, if F is the filling drawn in Figure 3, we have $\mathrm{ne}_2(F) = 6$ and $\mathrm{se}_2(F) = 4$, while for the filling in Figure 4 we have $\mathrm{ne}_2(F) = \mathrm{se}_2(F) = 4$. It is natural in view of the results presented in the introduction to ask if the statistics cros_2 and nest_2 are equidistributed over all simple graphs of $[n]$, or equivalently, if the statistics se_2 and ne_2 are equidistributed over $\mathcal{N}^{01}(\Delta_n)$ for any nonnegative integer n. More generally, one can ask if the statistics se_2 and ne_2 are equidistributed over $\mathcal{N}^{01}(T)$ for any moon polyomino T. The answer to these questions is no by means of Proposition 6.1. However, it appears that for particular 01-fillings we have such an equidistribution and even more, namely the symmetry of the distribution of the joint statistic $(\mathrm{ne}_2, \mathrm{se}_2)$. Let $\mathcal{N}^c(T)$ (resp., $\mathcal{N}^r(T)$) be the set of all 01-fillings of T with at most one 1 in each column (resp., row), and $\mathcal{N}(T) := \mathcal{N}^c(T) \cap \mathcal{N}^r(T)$ the set of all 01-fillings of T with at most one 1 in each column and in each row. Then our main result can be stated as follows.

Theorem 2.1. *For any moon polyomino T, the distribution of the joint statistic* $(\mathrm{ne}_2, \mathrm{se}_2)$ *over any $\mathcal{B} \in \{\mathcal{N}(T), \mathcal{N}^c(T), \mathcal{N}^r(T)\}$ is symmetric, or equivalently*

$$\sum_{F \in \mathcal{B}} p^{\mathrm{ne}_2(F)} q^{\mathrm{se}_2(F)} = \sum_{F \in \mathcal{B}} p^{\mathrm{se}_2(F)} q^{\mathrm{ne}_2(F)}. \tag{2.1}$$

In fact we have obtained much stronger results. Before stating these results, we need to introduce some definitions. Let T be a moon polyomino with s rows and t columns. By convention, we always label the rows of T from top to bottom in increasing order by $\{1, 2, \ldots, s\}$ and the columns of T from left to right in increasing order by $\{1, 2, \ldots, t\}$. The row labeled i and the column labeled j will be denoted respectively by R_i and C_j. The *length-row sequence* of T, denoted $r(T)$, is the sequence $(r_1, r_2, \ldots, r_s)$ where r_i is the length (i.e., the number of cells) of the row R_i. Similarly, the *length-column sequence* of T, denoted $c(T)$, is the sequence $(c_1, c_2, \ldots, c_t)$ where c_i is the length of the column C_i. Clearly, the length-row and length-column sequence of any moon polyomino are always unimodal sequences, that is there exist (unique) integers i_0 and j_0 such that $r_1 \le r_2 \le \cdots \le r_{i_0} > r_{i_0+1} \ge \cdots \ge r_s$ and $c_1 \le c_2 \le \cdots \le c_{j_0} > c_{j_0+1} \ge \cdots \ge c_t$. The *upper part of* T, denoted $Up(T)$, is the set of rows R_i with $1 \le i \le i_0$, and the *lower part*, denoted $Low(T)$, the set of rows R_i, $i_0 + 1 \le i \le s$. Similarly, the *left part* of T, denoted $Left(T)$, is the set of columns C_i with $1 \le i \le j_0$, and the *right part*, denoted $Right(T)$, the set of columns C_i, $j_0 + 1 \le i \le t$. For instance, if T is the moon polyomino in Figure 4, we have $r(T) = (4, 6, 9, 10, 10, 8, 5, 2)$, $Up(T) = \{R_i : 1 \le i \le 5\}$ and $Low(T) = \{R_6, R_7, R_8\}$, and $c(T) = (2, 3, 4, 4, 5, 7, 8, 8, 7, 6)$, $Left(T) = \{C_i : 1 \le i \le 8\}$ and $Right(T) = \{C_9, C_{10}\}$. Define the relation $\prec$ on the rows of T as follows: $R_i \prec R_j$ if and only if

- $r_i < r_j$, or
- $r_i = r_j$, $R_i \in Up(T)$ and $R_j \in Low(T)$, or

- $r_i = r_j$, $R_i, R_j \in Up(T)$ and R_i is above R_j, or
- $r_i = r_j$, $R_i, R_j \in Low(T)$ and R_i is below R_j.

Similarly, define the relation $\prec$ (for convenience, we use the same symbol than for rows) on the columns of T defined by $C_i \prec C_j$ if and only if

- $c_i < c_j$, or
- $c_i = c_j$, $C_i \in Left(T)$ and $C_j \in Right(T)$, or
- $c_i = c_j$, $C_i, C_j \in Left(T)$ and C_i is to the left of C_j, or
- $c_i = c_j$, $C_i, C_j \in Right(T)$ and C_i is to the right of C_j.

It is easy to check that the relation $\prec$ is a total order both on rows and columns of T. For instance, if T is the moon polyomino in Figure 4 we have $R_8 \prec R_1 \prec R_7 \prec R_2 \prec R_6 \prec R_3 \prec R_4 \prec R_5$ and $C_1 \prec C_2 \prec C_3 \prec C_4 \prec C_5 \prec C_{10} \prec C_6 \prec C_9 \prec C_7 \prec C_8$.

Let F be a 01-filling of T. A cell of F is said to be *empty* if it has been assigned the value 0. We also say that a row (resp., column) of F is empty if all its cells are empty. The indices of the empty rows and columns of F are denoted $\mathrm{ER}(F)$ and $\mathrm{EC}(F)$, respectively. For instance if F is the 01-filling given in Figure 4, then $\mathrm{ER}(F) = \{3, 7\}$ and $\mathrm{EC}(F) = \{3, 10\}$.

Given a s-uple $\mathbf{m} = (m_1, \ldots, m_s)$ of positive integers and A a subset of $[t]$, we denote by $\mathcal{N}^c(T, \mathbf{m})$ the set of 01-fillings in $\mathcal{N}^c(T)$ with exactly m_i 1's in row R_i and by $\mathcal{N}^c(T, \mathbf{m}; A)$ the set of fillings F in $\mathcal{N}^c(T, \mathbf{m})$ such that $\mathrm{EC}(F) = A$. For instance, the filling F given in Figure 4 belong to $\mathcal{N}^c(T, \mathbf{m}; A)$, with $\mathbf{m} = (1, 2, 0, 1, 2, 1, 0, 1)$ and $A = \{3, 10\}$. Similarly, given a t-uple $\mathbf{n} = (n_1, \ldots, n_t)$ of positive integers and a subset B of $[s]$, we denote by $\mathcal{N}^r(T, \mathbf{n})$ the set of 01-fillings in $\mathcal{N}^r(T)$ with exactly n_i 1's in column C_i and by $\mathcal{N}^r(T, \mathbf{n}; B)$ the set of fillings F in $\mathcal{N}^r(T, \mathbf{n})$ such that $\mathrm{ER}(F) = B$. Also, for A, B two subsets of $[t]$ and $[s]$ respectively, we denote by $\mathcal{N}(T; A, B)$ the set of 01-fillings in $\mathcal{N}(T)$ such that $\mathrm{EC}(F) = A$ and $\mathrm{ER}(F) = B$.

For positive integers n and k, let $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$ be the p, q-Gaussian coefficient defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \begin{cases} \frac{[n]_{p,q}!}{[k]_{p,q}!\,[n-k]_{p,q}!}, & \text{if } 0 \le k \le n; \\ 0, & \text{otherwise.} \end{cases}$$

where, as usual in p, q-theory, the p, q-integer $[r]_{p,q}$ is given by

$$[r]_{p,q} := \frac{p^i - q^i}{p - q} = (p^{i-1} + p^{i-2}q + \cdots + p^j q^{i-j-1} + \cdots + pq^{i-2} + q^{i-1}),$$

and the p, q-factorial $[r]_{p,q}!$ by $[r]_{p,q}! := \prod_{i=1}^{r} [i]_{p,q}$.

Let T be a moon polyomino with s rows and t columns, $\mathbf{m} = (m_1, \ldots, m_s)$ a s-uple of positive integers, $\mathbf{n} = (n_1, \ldots, n_t)$ a t-uple of positive integers, and A, B two subsets of $[t]$ and $[s]$ respectively. Suppose $R_{i_1} \prec R_{i_2} \prec \cdots \prec R_{i_s}$ and $C_{j_1} \prec C_{j_2} \prec \cdots \prec C_{j_t}$. Then for $u \in [s]$ and $v \in [t]$, define h_{i_u} and h'_{j_v} by

$$h_{i_u} = r_{i_u} - (m_{i_1} + m_{i_2} + \cdots + m_{i_{u-1}}) - a_{i_u}, \tag{2.2}$$

$$h'_{j_v} = c_{j_v} - (n_{j_1} + n_{j_2} + \cdots + n_{j_{v-1}}) - b_{j_v}, \tag{2.3}$$

where r_{i_u} is the length of the row R_{i_u} and a_{i_u} is the number of indices $k \in A$ such that the column C_k intersect the row R_{i_u}, and c_{j_v} is the length of the column C_{j_v} and b_{j_v} is the number of indices $k \in B$ such that the row R_k intersect the column C_{j_v}.

The following result gives the distributions of the joint statistic $(\mathrm{ne}_2, \mathrm{se}_2)$ over $\mathcal{N}^c(T, \mathbf{m}; A)$ and $\mathcal{N}^r(T, \mathbf{n}; B)$.

Theorem 2.2. *For any moon polyomino T with s rows and t columns, the distributions of the joint statistic $(\mathrm{ne}_2, \mathrm{se}_2)$ over each $\mathcal{N}^c(T, \mathbf{m}; A)$ and $\mathcal{N}^r(T, \mathbf{n}; B)$ are given by*

$$\sum_{F \in \mathcal{N}^c(T,\mathbf{m};A)} p^{\mathrm{ne}_2(F)} q^{\mathrm{se}_2(F)} = \prod_{d=1}^{s} \begin{bmatrix} h_d \\ m_d \end{bmatrix}_{p,q}, \tag{2.4}$$

$$\sum_{F \in \mathcal{N}^r(T,\mathbf{n};B)} p^{\mathrm{ne}_2(F)} q^{\mathrm{se}_2(F)} = \prod_{d=1}^{t} \begin{bmatrix} h'_d \\ n_d \end{bmatrix}_{p,q}, \tag{2.5}$$

where h_d and h'_d are defined by (2.2) *and* (2.3).

As an immediate consequence (take $\mathbf{m} \in \{0,1\}^s$ or $\mathbf{n} \in \{0,1\}^t$ in Theorem 2.2), we obtain the following result.

Corollary 2.3. *For any moon polyomino T with s rows and t columns, the distribution of the joint statistic $(\mathrm{ne}_2, \mathrm{se}_2)$ over $\mathcal{N}(T; A, B)$ is given by*

$$\sum_{F \in \mathcal{N}(T;A,B)} p^{\mathrm{ne}_2(F)} q^{\mathrm{se}_2(F)} = \prod_{d \in [s] \setminus B} [h_d]_{p,q} = \prod_{d \in [t] \setminus A} [h'_d]_{p,q}, \tag{2.6}$$

where h_d and h'_d are defined by (2.2) *and* (2.3).

It is not difficult to show that (2.4) and (2.5) are equivalent. Indeed, if T is a moon polyomino with s rows and t columns, then the arrangement of cells T^t obtained from T by rotation about 90° is also a moon polyomino but with t rows and s columns. Moreover it is obvious that the application $F \mapsto F^t$ which associates to each filling F of T the filling F^t of T^t obtained from F by rotation about 90° establishes a bijection from $\mathcal{N}^c(T, \mathbf{m}; A)$ onto $\mathcal{N}^r(T^t, \mathbf{m}; A)$ which sends the joint statistic $(\mathrm{ne}_2, \mathrm{se}_2)$ onto $(\mathrm{se}_2, \mathrm{ne}_2)$ for any $\mathbf{m}$ and A. See Figure 5 for an illustration.

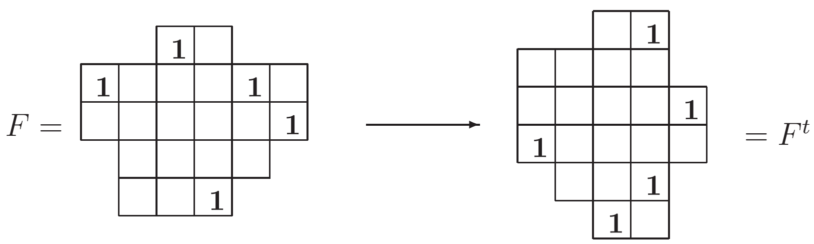


FIGURE 5. Rotation about 90° for filling.

Since the p,q-integer $[n]_{p,q}$ is symmetric in the variables p and q for any positive integer n, we have the following results.

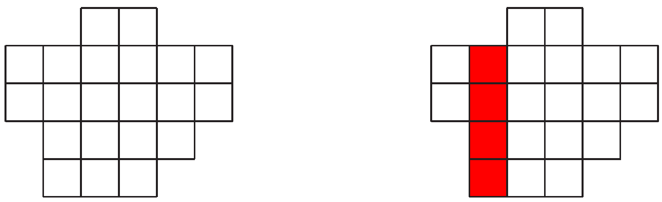
Corollary 2.4. *For any moon polyomino T, the joint statistic* $(\mathrm{ne}_2, \mathrm{se}_2)$ *is symmetrically distributed over each* $\mathcal{N}^c(T, \mathbf{m}; A)$, $\mathcal{N}^r(T, \mathbf{n}; B)$ *and* $\mathcal{N}(T; A, B)$. *In particular, it is symmetrically distributed over each* $\mathcal{N}^c(T, \mathbf{m})$ *and* $\mathcal{N}^r(T, \mathbf{n})$.

For any positive integer k denote by $\mathcal{N}^c(T; k)$, $\mathcal{N}^r(T; k)$ and $\mathcal{N}(T; k)$ the set of 01-fillings in $\mathcal{N}^c(T)$, $\mathcal{N}^r(T)$ and $\mathcal{N}(T)$ with exactly k ones, respectively.

Corollary 2.5. *For any moon polyomino T and positive integer k, the joint statistic* $(\mathrm{ne}_2, \mathrm{se}_2)$ *is symmetrically distributed over each* $\mathcal{N}^c(T; k)$, $\mathcal{N}^r(T; k)$ *and* $\mathcal{N}(T; k)$. *Summing over all positive integers k, we recover Theorem 2.1.*

The paper is organized as follows. In Section 3, we show briefly how the results on the symmetry of the joint statistic $(\mathrm{cros}_2, \mathrm{nest}_2)$ presented in the introduction can be obtained from the above results. In section 4, we prove Theorem 2.1 and in Section 5, we present a bijective proof of Corollary 2.4. Finally, we conclude this paper with some remarks and problems.

We end this section by illustrating Theorem 2.2. Suppose T is the moon polyomino given below and $A = \{2\}$.



Then we have:

- $R_{i_1} \prec R_{i_2} \prec R_{i_3} \prec R_{i_4} \prec R_{i_5}$ with $i_1 = 1$, $i_2 = 5$, $i_3 = 4$, $i_4 = 2$, $i_5 = 3$.
- The column labeled 2 intersect the rows labeled $2, 3, 4, 5$, thus $a_1 = 0$, $a_2 = a_3 = a_4 = a_5 = 1$.

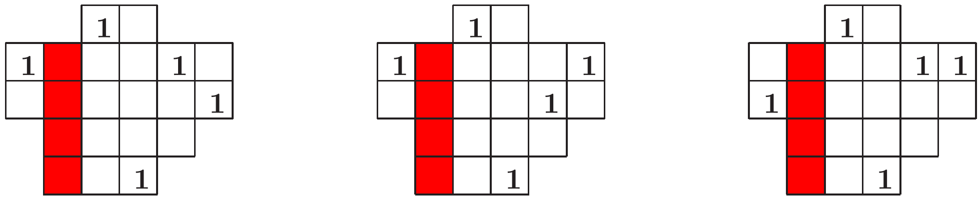
Suppose $\mathbf{m} = (1, 2, 1, 0, 1)$. We then have

$$\begin{aligned}
h_{i_1} &= h_1 = r_1 - a_1 = 2,\\
h_{i_2} &= h_5 = r_5 - m_1 - a_5 = 1,\\
h_{i_3} &= h_4 = r_4 - (m_1 + m_5) - a_4 = 1,\\
h_{i_4} &= h_2 = r_2 - (m_1 + m_5 + m_4) - a_2 = 3,\\
h_{i_5} &= h_3 = r_3 - (m_1 + m_5 + m_4 + m_2) - a_3 = 1.
\end{aligned}$$

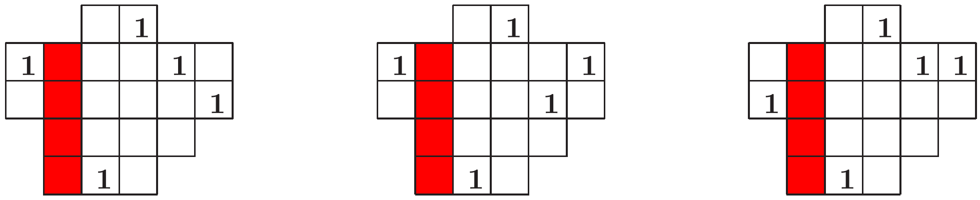
It then follows from Theorem 2.2 that

$$\sum_{F \in \mathcal{N}^c(T, \mathbf{m}; A)} p^{\mathrm{ne}_2(F)} q^{\mathrm{se}_2(F)} = \prod_{j=1}^{5} \begin{bmatrix} h_j \\ m_j \end{bmatrix}_{p,q} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{p,q} \begin{bmatrix} 3 \\ 2 \end{bmatrix}_{p,q} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{p,q} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{p,q} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{p,q} = p^3 + 2p^2q + 2pq^2 + q^3.$$

On the other hand, the fillings in $\mathcal{N}^c(T, \mathbf{m}; A)$ and the corresponding values of ne_2 and se_2 are listed below.



$\mathrm{ne}_2 = 0\ ,\ \mathrm{se}_2 = 3$ $\quad$ $\mathrm{ne}_2 = 1\ ,\ \mathrm{se}_2 = 2$ $\quad$ $\mathrm{ne}_2 = 2\ ,\ \mathrm{se}_2 = 1$



$\mathrm{ne}_2 = 1\ ,\ \mathrm{se}_2 = 2$ $\quad$ $\mathrm{ne}_2 = 2\ ,\ \mathrm{se}_2 = 1$ $\quad$ $\mathrm{ne}_2 = 3\ ,\ \mathrm{se}_2 = 0$

Summing up we get $\sum_{F\in\mathcal{N}^c(T,\mathbf{m};A)} p^{\mathrm{ne}_2(F)}q^{\mathrm{se}_2(F)} = p^3 + 2p^2q + 2pq^2 + q^3$, as desired.

3. Symmetry of 2-crossings and 2-nestings in linked partitions, set partitions and matchings

In this section we show briefly how results on the enumeration of 2-crossings and 2-nestings can be recovered from the results obtained in this paper. Let G be a simple graph on $[n]$. The multiset of lefthand (resp., righthand) endpoints of the arcs of G will be denoted by left(G) (resp., right(G)). For instance, if G is the graph drawn in Figure 3, we have left(G) $= \{1, 2, 2, 3, 5, 6, 6, 9\}$ and right(G) $= \{3, 4, 6, 7, 9, 9, 10, 11\}$. For S and T two multisubsets of $[n]$, we will denote by $\mathcal{G}_n(S, T)$ the set of simple graphs G on $[n]$ satisfying left(G) = S and right(G) = T.

Suppose F is the 01-filling of the triangular shape Δ_n which corresponds to the graph G. For convenience, we joined an empty column at the right and an empty row at the top of Δ_n, and columns are labeled from left to right and rows from top to bottom by $\{1, 2, \ldots, n\}$. It is obvious that the number of 1's in the column (resp., row) labeled i is equal to the multiplicity of i in left(G) (resp., right(G)). See Figure 3 for an illustration. Taking the moon polyomino $T := \Delta_n$ in Theorem 2.2, we obtain the following results.

Let (S, T) be a pair of multisubsets of $[n]$ and denote by m_i the multiplicity of i in T and by m'_i the multiplicity of i in S. Also for any $i \in T$ set $h_i = |\{j \in S \,|\, j < i\}| - |\{j \in T \,|\, j < i\}|$ and for any $i \in S$ set $h'_i = |\{j \in T \,|\, j > i\}| - |\{j \in S \,|\, j > i\}|$.

Corollary 3.1. *Let (S, T) be a pair of multisubsets of $[n]$.*

(1) *If all elements of S have multiplicity 1, then*

$$\sum_{G\in\mathcal{G}_n(S,T)} p^{\mathrm{ne}_2(F)}q^{\mathrm{se}_2(F)} = \sum_{G\in\mathcal{G}_n(S,T)} p^{\mathrm{se}_2(F)}q^{\mathrm{ne}_2(F)} = \prod_{i\in T} \begin{bmatrix} h_i \\ m_i \end{bmatrix}_{p,q}. \tag{3.1}$$

(2) *If all elements of T have multiplicity* 1, *then*

$$\sum_{G\in \mathcal{G}_n(S,T)} p^{\mathrm{ne}_2(F)}q^{\mathrm{se}_2(F)} = \sum_{G\in \mathcal{G}_n(S,T)} p^{\mathrm{se}_2(F)}q^{\mathrm{ne}_2(F)} = \prod_{i\in S} \begin{bmatrix} h'_i \\ m'_i \end{bmatrix}_{p,q}. \tag{3.2}$$

In particular, the joint statistic $(\mathrm{cros}_2, \mathrm{nest}_2)$ *is symmetrically distributed over* $\mathcal{G}_n(S,T)$ *if*

- *either all elements of S have multiplicity* 1,
- *either all elements of T have multiplicity* 1.

Note that (3.2) is equivalent to a result of Chen et al. [2, Theorem 3.5] on the enumeration of 2-crossings and 2-nestings in linked set partitions. Let E and F be two finite sets of nonnegative integers. We say that E and F are *nearly disjoint* if for every $i \in E\cap F$, one of the following holds:

(a) $i = \min(E)$, $|E| > 1$ and $i \neq \min(F)$, or
(b) $i = \min(F)$, $|F| > 1$ and $i \neq \min(E)$.

A *linked partition* (see [2]) of $[n]$ is a collection of non-empty and pairwise nearly disjoint subsets whose union is $[n]$. The set of all linked partitions of $[n]$ will be denoted by $\mathcal{LP}_n$. The linear representation G_π of a linked partition $\pi \in \mathcal{LP}_n$ is the graph on $[n]$ where i and j are connected by an arc if and only if j lies in a block B with $i = \mathrm{Min}(B)$. An illustration is given in Figure 6. Clearly, the map $\pi \mapsto G_\pi$ establishes a bijection

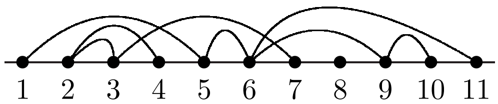


FIGURE 6. Linear representation of $\pi = \{1,5\}\{2,3,4\}\{3,7\}\{5,6\}\{6,9,11\}\{8\}\{9,10\}$

between linked set partitions and simple graphs G such that all elements of right(G) have multiplicity one. For $S,T \subseteq [n]$ two multisubsets of $[n]$, denote by $\mathcal{LP}_n(S,T)$ the set $\{\pi \in \mathcal{LP}_n : \mathrm{left(G_\pi) = S}\,,\, \mathrm{right(G_\pi) = T}\}$. Then (3.2) can be rewritten

$$\sum_{\pi\in \mathcal{LP}_n(S,T)} p^{\mathrm{cros}_2(G_\pi)}q^{\mathrm{nest}_2(G_\pi)} = \sum_{\pi\in \mathcal{LP}_n(S,T)} p^{\mathrm{nest}_2(G_\pi)}q^{\mathrm{cros}_2(G_\pi)} = \prod_{i\in S} \begin{bmatrix} h'_i \\ m'_i \end{bmatrix}_{p,q}, \tag{3.3}$$

where h'_i and m'_i are defined as in Corollary (3.1), which is equivalent to a result of Chen et al. [2, Theorem 3.5].

Now consider the map $\pi \mapsto St_\pi$ which associates to each set partition its standard representation (see Figure 2). Clearly, this map establishes a bijection between set partitions and simple graphs G such that all elements of right(G) and left(G) have multiplicity one. Applying Corollary 3.1 with $m_i = 1$ (or (3.3) with $m'_i = 1$) we recover (1.6) and the following identity which is implicit in [9, Section 4]

$$\sum_{\pi\in \mathcal{P}_n(S,T)} p^{\mathrm{cros}_2(\pi)}q^{\mathrm{nest}_2(\pi)} = \sum_{\pi\in \mathcal{P}_n(S,T)} p^{\mathrm{nest}_2(\pi)}q^{\mathrm{cros}_2(\pi)} = \prod_{i\in \mathcal{O}} [h_i]_{p,q} = \prod_{i\in T} [h'_i]_{p,q}, \tag{3.4}$$

where h_i and h'_i are defined as in Corollary (3.1).

4. Proof of Theorem 2.2

As explained in Section 2, it suffices to prove the first part of Theorem 2.2 that is the identity (2.4). Throughout this section, T is a moon polyomino with s rows and length-row sequence $(r_1, r_2, \ldots, r_s)$, and $\mathbf{m} = (m_1, \ldots, m_s)$ is a s-uple of positive integers.

4.1. **Preliminaries.** Let i, $1 \leq i \leq s$, be an integer. The *i-th rectangle of T*, is the greatest rectangle contained in T whose top (resp., bottom) row is R_i if $R_i \in Up(T)$ (resp., $R_i \in Low(T)$). An illustration is given in Figure 7.

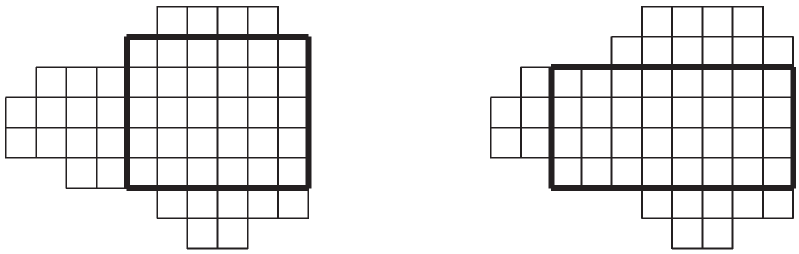


Figure 7. *left*: the 2-th rectangle, *right*: the 6-th rectangle.

Let F be a 01-filling of T in $\mathcal{N}^c(T, \mathbf{m})$. The *coloring of F* is the colored filling obtained from F by:

- coloring the cells of the empty columns,
- for $i = 1, \ldots, s$, coloring the cells which are both contained in the i-th rectangle and
  - if $R_i \in Up(T)$, below a 1 of R_i.
  - if $R_i \in Low(T)$, above a 1 of R_i.

An illustration is given in Figure 8. Throughout this paper, we identify a filling with its coloring. For instance, "the cell c of the filling F is uncolored" means that "the cell c is uncolored in the coloring of F",...

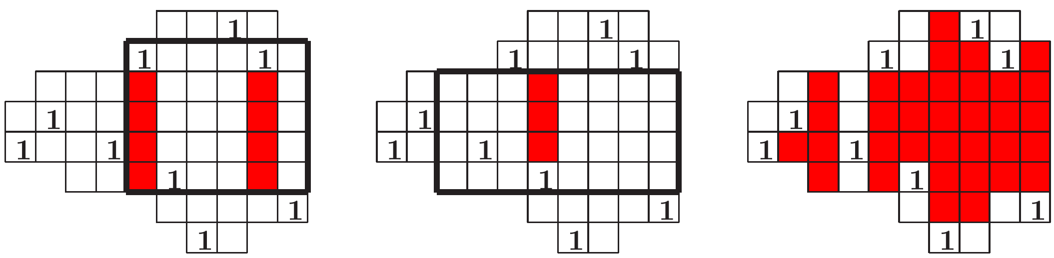


Figure 8. *left*: coloring induced by R_2, *center*: coloring induced by R_6, *right*: full coloring.

The interest of coloring a 01-filling is in the following result. Let c be a cell of F. If c contains a 1 denote by $\mathrm{luc}(c; F)$ (resp., $\mathrm{ruc}(c; F)$) the numbers of uncolored empty cells which are both to the left (resp., right) and in the same row than the cell c in F. If c is empty, set $\mathrm{luc}(c;F)=\mathrm{ruc}(c;F)=0$.

Proposition 4.1. *Let $F \in \mathcal{N}^c(T)$ and c be a cell of R_i containing a 1. Then* $\mathrm{luc}(c;F)$ *(resp.,* $\mathrm{ruc}(c;F)$*) is equal to*

- *if $R_i \in Up(T)$: the number of ascents (resp., descents) contained in the i-th rectangle of F whose North-east (resp., North-west)* 1 *is in c,*
- *if $R_i \in Low(T)$: the number of descents (resp. ascents) contained in the i-th rectangle of F whose South-east (resp., South-west)* 1 *is in c.*

Sketch of the proof of Proposition 4.1. Suppose $R_i \in Up(T)$ and let c be a cell of R_i containing a 1.

Let c' be an empty uncolored cell in R_i to the left (resp., right) of c. Suppose c' belong to the column C_k. By the definition of the coloring of polyominoes, the column C_k contains a 1 (otherwise all the cells of C_k, in particular c', would be colored). Moreover, the cell c'' of C_k containing a 1 must belong to a row R_j with $R_i \prec R_j$ (otherwise all the cells of C_k in the i-th rectangle of T, in particular c', would be colored), and thus c'' belong to the i-th rectangle. Since $R_i \in Up(T)$, the row R_i is the top row of the i-th rectangle of T, and thus the cell c'' is to the South-west (resp., South-east) of the cell c. Finally, the sequence $c''c$ (resp., cc'') is an ascent (resp., a descent) contained in the i-th rectangle of F.

Reversely, let c'' be a cell of F such that the pair $c''c$ (resp., cc'') is an ascent (resp., descent) of F contained in the i-th rectangle of F. Suppose c'' belong to C_k and let c' be the cell of F at the intersection of the column C_k and the row R_i. Clearly, C_k is empty (there is at most one 1 in each column). It remains to show that the cell c' is uncolored. This follows from the fact that c'' belong to a row R_j with $R_i \prec R_j$ (since c'' is in the i-th rectangle). We thus have proved the first part of Proposition 4.1.

The second part can be proved by a similar reasoning. Therefore, the details are left to the reader.

□

Note that Proposition 4.1 lead to the following decompositions of ne_2 and se_2:

$$\mathrm{ne}_2(F) = \sum_{c\,\in\,Up(F)} \mathrm{luc}(c;F) + \sum_{c\,\in\,Low(F)} \mathrm{ruc}(c;F), \tag{4.1}$$

$$\mathrm{se}_2(F) = \sum_{c\,\in\,Up(F)} \mathrm{ruc}(c;F) + \sum_{c\,\in\,Low(F)} \mathrm{luc}(c;F). \tag{4.2}$$

4.2. **A correspondence between 01-fillings and sequence of compositions.** If n and k are positive integers, we will denote by $\mathcal{C}_k(n)$ the set of compositions of n into k positive parts. Recall that a element in $\mathcal{C}_k(n)$ is just a k-uple $(b_1, b_2, \ldots, b_k)$ of positive integers such that $b_1+b_2+\cdots+b_k=n$. The proof of Theorem 2.2 is based on a bijection

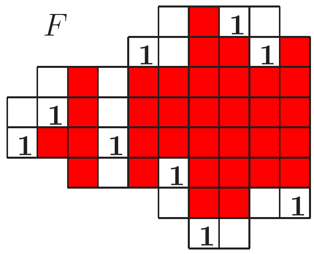
$$\Psi : \mathcal{N}^c(T,\mathbf{m};A) \to \mathcal{C}_{m_1+1}(h_1-m_1)\times\mathcal{C}_{m_2+1}(h_2-m_2)\times\cdots\times\mathcal{C}_{m_s+1}(h_s-m_s)$$

which keeps track of the statistics ne_2 and se_2.

Algorithm for Ψ. For $F \in \mathcal{N}^c(T, \mathbf{m}; A)$ associate the sequence of compositions $\Psi(F) := (\mathrm{c}^{(1)}, \mathrm{c}^{(2)}, \ldots, \mathrm{c}^{(s)})$, where for $i = 1, \ldots, s$, the composition $\mathrm{c}^{(i)}$ is defined by

- $\mathrm{c}^{(i)} = (0)$ if $m_i = 0$, otherwise
- $\mathrm{c}^{(i)} = (\mathrm{c}_1^{(i)}, \ldots, \mathrm{c}_{m_i+1}^{(i)})$ where $\mathrm{c}_1^{(i)}$ (resp., $\mathrm{c}_j^{(i)}$ for $2 \le j \le m_i$, $\mathrm{c}_{m_i+1}^{(i)}$) is the number of uncolored cells in R_i to the left of the first 1 (resp., between the j-th 1 and the $(j+1)$-th 1, to the right of the last 1) of R_i in the coloring of F.

An illustration is given in Figure 9.



$$\xrightarrow{\Psi}\qquad \begin{aligned} \mathrm{c}^{(1)} &= (1,1)\\ \mathrm{c}^{(2)} &= (0,1,0)\\ \mathrm{c}^{(3)} &= (0)\\ \mathrm{c}^{(4)} &= (1,1)\\ \mathrm{c}^{(5)} &= (0,0,0)\\ \mathrm{c}^{(6)} &= (1,0)\\ \mathrm{c}^{(7)} &= (2,0)\\ \mathrm{c}^{(8)} &= (0,1) \end{aligned}$$

FIGURE 9. The mapping Ψ

In order to show that Ψ is bijective, we describe its reverse. Let $\mathbf{c} = (\mathrm{c}^{(1)}, \mathrm{c}^{(2)}, \ldots, \mathrm{c}^{(s)})$ in $\mathcal{C}_{m_1+1}(h_1 - m_1) \times \mathcal{C}_{m_2+1}(h_2 - m_2) \times \cdots \times \mathcal{C}_{m_s+1}(h_s - m_s)$. Then define the 01-filling $\Upsilon(\mathbf{c})$ of T by the following process.

(1) Color the columns indexed by the set A of the polyomino T. Denote by F_0 the colored polyomino obtained.

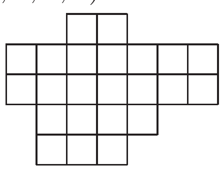
(2) Construct a sequence of colored fillings $(F_j)_{j=1\ldots s}$ of T as follows. Suppose $R_{i_1} \prec R_{i_2} \prec \cdots \prec R_{i_s}$. Then for j from 1 to s, the (colored) filling F_j is obtained from F_{j-1} as follows:

- if $m_{i_j} = 0$, do nothing,
- else, insert m_{i_j} 1's in the i_j-th row of F_{j-1} in such a way that the number of uncolored cells strictly
  - to the left of the first 1 is $\mathrm{c}_1^{(i_j)}$,
  - between the u-th 1 and the $(u+1)$-th 1, $1 \le u \le m_{i_j} - 1$, is $\mathrm{c}_{u+1}^{(i_j)}$,
  - to the right of the last 1 is $\mathrm{c}_{m_{i_j}+1}^{(i_j)}$.

  Next, color the cells which are both below (resp., above) the new 1's inserted in R_{i_j} and contained in the i_j-th rectangle if $R_{i_j} \in Up(T)$ (resp., $R_{i_j} \in Low(T)$).

(3) Set $\Upsilon(\mathbf{c}) = F_s$.

For a better understanding, we give an example. Suppose T is the moon polyomino given below, $A = \{2\}$ and $\mathbf{m} = (1, 2, 1, 0, 1)$. Note that $R_1 \prec R_5 \prec R_4 \prec R_2 \prec R_3$.



Suppose $\mathbf{c} = (\mathrm{c}^{(1)}, \mathrm{c}^{(2)}, \mathrm{c}^{(3)}, \mathrm{c}^{(4)}, \mathrm{c}^{(5)})$ with $\mathrm{c}^{(1)} = (1, 0)$, $\mathrm{c}^{(2)} = (1, 0, 1)$, $\mathrm{c}^{(3)} = (0, 0, 0)$, $\mathrm{c}^{(4)} = (0)$ and $\mathrm{c}^{(5)} = (0, 0)$. The step by step construction of $\Upsilon(\mathbf{c})$ is given in Figure 10. It is easily seen that Υ is the reverse of Ψ, and thus Ψ is bijective. Now let

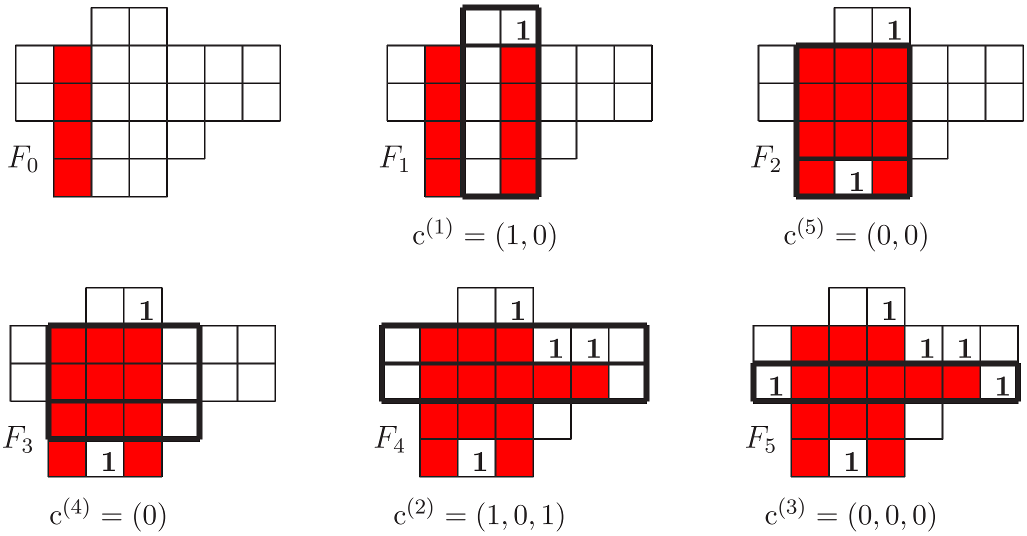


FIGURE 10. The step-by-step construction of $\Upsilon(\mathbf{c})$

$\mathbf{c} = (\mathrm{c}^{(1)}, \mathrm{c}^{(2)}, \ldots, \mathrm{c}^{(s)}) \in \mathcal{C}_{m_1+1}(h_1 - m_1) \times \mathcal{C}_{m_2+1}(h_2 - m_2) \times \cdots \times \mathcal{C}_{m_s+1}(h_s - m_s)$ and $F = \Upsilon(\mathbf{c})$ be the corresponding 01-filling. Let i be an integer in $[s]$ and ce be the cell of the i-th row R_i of F which contains the j-th 1 of R_i (from left to right as usual). It then follows from the definition of Υ that

$$\mathrm{luc}(ce; F) = \mathrm{c}_1^{(i)} + \mathrm{c}_2^{(i)} + \cdots + \mathrm{c}_j^{(i)} \quad \text{and} \quad \mathrm{ruc}(ce; F) = \mathrm{c}_{j+1}^{(i)} + \mathrm{c}_{j+2}^{(i)} + \cdots + \mathrm{c}_{m_i+1}^{(i)}.$$

We summarize the main properties of Ψ in Theorem 4.2.

Theorem 4.2. *The map $\Psi : \mathcal{N}^c(T, \mathbf{m}; A) \to \mathcal{C}_{m_1+1}(h_1 - m_1) \times \cdots \times \mathcal{C}_{m_s+1}(h_s - m_s)$ is a bijection such that for any $F \in \mathcal{N}^c(T, \mathbf{m}; A)$ and any cell ce in F, if $\Psi(F) = (\mathrm{c}^{(1)}, \mathrm{c}^{(2)}, \cdots, \mathrm{c}^{(s)})$, we have*

$$\mathrm{luc}(ce; F) = \mathrm{c}_1^{(i)} + \mathrm{c}_2^{(i)} + \cdots + \mathrm{c}_j^{(i)}$$
$$\mathrm{ruc}(ce; F) = \mathrm{c}_{j+1}^{(i)} + \mathrm{c}_{j+2}^{(i)} + \cdots + \mathrm{c}_{m_i+1}^{(i)}.$$

4.3. **Proof of** (2.4)**.** The proof is based on the correspondence Ψ and the following identity, which can be easily proved (by induction for instance).

Lemma 4.3. *For any integers $n \geq k \geq 0$,*

$$\sum_{(\mathrm{c}_1, \mathrm{c}_2, \ldots, \mathrm{c}_{k+1}) \in \mathcal{C}_{k+1}(n)} \prod_{j=1}^{k} p^{\sum_{r=1}^{j} \mathrm{c}_r} q^{\sum_{r=j+1}^{k+1} \mathrm{c}_r} = \begin{bmatrix} n+k \\ k \end{bmatrix}_{p,q}.$$

Suppose $Up(T) = \{R_1, R_2, \ldots, R_{i_0}\}$ and $Low(T) = \{R_{i_0+1}, \ldots, R_s\}$. By (4.1) and (4.2) we have

$$\sum_{F\in\mathcal{N}^c(T,\mathbf{m};A)} p^{\mathrm{ne}_2(F)}q^{\mathrm{se}_2(F)} = \sum_{F\in\mathcal{N}^c(T,\mathbf{m};A)} \prod_{ce\in Up(T)} p^{\mathrm{luc}(ce;F)}q^{\mathrm{ruc}(ce;F)} \prod_{ce\in Low(T)} p^{\mathrm{ruc}(ce;F)}q^{\mathrm{luc}(ce;F)}$$

$$= \sum_{F\in\mathcal{N}^c(T,\mathbf{m};A)} \prod_{i=1}^{i_0}\prod_{ce\in R_i} p^{\mathrm{luc}(ce;F)}q^{\mathrm{ruc}(ce;F)} \prod_{i=i_0+1}^{s}\prod_{ce\in R_i} p^{\mathrm{ruc}(ce;F)}q^{\mathrm{luc}(ce;F)}.$$

Let $\mathcal{C} = \mathcal{C}_{m_1+1}(h_1-m_1)\times\cdots\times\mathcal{C}_{m_s+1}(h_s-m_s)$. It follows from Theorem 4.2 that the right-hand side of the last equality can be rewritten

$$\sum_{(\mathrm{c}^{(1)},\mathrm{c}^{(2)},\ldots,\mathrm{c}^{(s)})\in\mathcal{C}} \prod_{i=1}^{i_0}\left(\prod_{j=1}^{m_i} p^{\sum_{r=1}^{j}\mathrm{c}_r^{(i)}} q^{\sum_{r=j+1}^{m_i+1}\mathrm{c}_r^{(i)}}\right) \prod_{i=i_0+1}^{s}\left(\prod_{j=1}^{m_i} p^{\sum_{r=j+1}^{m_i+1}\mathrm{c}_r^{(i)}} q^{\sum_{r=1}^{j}\mathrm{c}_r^{(i)}}\right)$$

$$=\prod_{i=1}^{i_0}\left(\sum_{\mathrm{c}^{(i)}\in\mathcal{C}_{m_i+1}(h_i-m_i)} \prod_{j=1}^{m_i} p^{\sum_{r=1}^{j}\mathrm{c}_r^{(i)}} q^{\sum_{r=j+1}^{m_i+1}\mathrm{c}_r^{(i)}}\right) \prod_{i=i_0+1}^{s}\left(\sum_{\mathrm{c}^{(i)}\in\mathcal{C}_{m_i+1}(h_i-m_i)} \prod_{j=1}^{m_i} p^{\sum_{r=j+1}^{m_i+1}\mathrm{c}_r^{(i)}} q^{\sum_{r=1}^{j}\mathrm{c}_r^{(i)}}\right).$$

Applying Lemma 4.3 conclude the proof of (2.4) and thus of Theorem 2.2.

5. A bijective proof of Corollary 2.4

Let T be a moon polyomino with s rows and t columns. In this section, we present a mapping Φ such that for any s-uple of positive integers $\mathbf{m} = (m_1, m_2, \ldots, m_s)$ and set A of positive integers, the map Φ is a bijection $\Phi : \mathcal{N}^c(T, \mathbf{m}; A) \to \mathcal{N}^c(T, \mathbf{m}; A)$ such that for any $F \in \mathcal{N}^c(T, \mathbf{m}; A)$, we have

$$(\mathrm{ne}_2, \mathrm{se}_2)(\Phi(F)) = (\mathrm{se}_2, \mathrm{ne}_2)(F).$$

This gives a direct combinatorial proof of the symmetry of the joint distribution of $(\mathrm{ne}_2, \mathrm{se}_2)$ over each $\mathcal{N}^c(T, \mathbf{m}; A)$, $\mathcal{N}(T; A, B)$ (set $m_i = 0$ for $i \in B$ and 1 otherwise), and $\mathcal{N}^r(T, \mathbf{n}; B)$ (compose with the rotation about 90°).

In fact, the map Φ is just a byproduct of the constructions given in the previous section. It is also a generalization of an involution presented in [9] to prove the symmetry of $(\mathrm{cros}_2, \mathrm{nest}_2)$ over set partitions and matchings.

Let $\mathrm{c} = (\mathrm{c}_1, \mathrm{c}_2, \ldots, \mathrm{c}_k)$ be a composition. Define the *reverse* $\mathrm{rev}(\mathrm{c})$ of c as the composition $\mathrm{rev}(\mathrm{c}) = (\mathrm{c}_k, \mathrm{c}_{k-1}, \ldots, \mathrm{c}_1)$. Given a sequence of compositions $\mathrm{c} = (\mathrm{c}^{(1)}, \mathrm{c}^{(2)}, \ldots, \mathrm{c}^{(s)})$, we set $\mathrm{Rev}(\mathrm{c}) = (\mathrm{rev}(\mathrm{c}^{(1)}), \mathrm{rev}(\mathrm{c}^{(2)}), \ldots, \mathrm{rev}(\mathrm{c}^{(s)}))$.

Let $\Phi : \mathcal{N}^c(T, \mathbf{m}; A) \mapsto \mathcal{N}^c(T, \mathbf{m}; A)$ be the map defined by

$$\Phi = \Psi^{-1} \circ \mathrm{Rev} \circ \Psi = \Upsilon \circ \mathrm{Rev} \circ \Psi.$$

The following proposition is an immediate consequence of the properties of Ψ (see Theorem 4.2).

Proposition 5.1. *The map Φ is an involution on $\mathcal{N}^c(T, \mathbf{m}; A)$ such that for any $F \in \mathcal{N}^c(T, \mathbf{m}; A)$ and any cell c of T, we have*

$$\mathrm{luc}(c; \Phi(F)) = \mathrm{ruc}(c; F) \quad \textit{and} \quad \mathrm{ruc}(c; \Phi(F)) = \mathrm{luc}(c; F).$$

In particular, we have $(\mathrm{ne}_2, \mathrm{se}_2)(\Phi(F)) = (\mathrm{se}_2, \mathrm{ne}_2)(F)$.

It could be useful to give a direct description of Φ. Let $F \in \mathcal{N}^c(T, \mathbf{m}; A)$.

(1) Color the columns of polyomino T indexed by the set A. Denote by F'_0 the colored polyomino obtained.

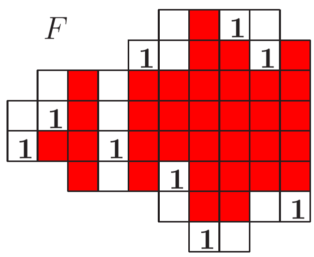
(2) Contruct a sequence of colored fillings $(F'_j)_{j=1\ldots s}$ of T as follows. Suppose $R_{i_1} \prec R_{i_2} \prec \cdots \prec R_{i_s}$. Then for j from 1 to s, the (colored) filling F'_j is obtained from F'_{j-1} as follows:

- if $m_{i_j} = 0$, do nothing,
- else, read the m_{i_j} 1's in the i_j-th row of F from left to right and denote the number of uncolored cells (in the coloring of F) strictly
  - to the left of the first 1 by t_0,
  - between the u-th 1 and the $(u+1)$-th 1, $1 \le u \le m_{i_j} - 1$, by t_u,
  - to the right of the last 1 by $t_{m_{i_j}}$.

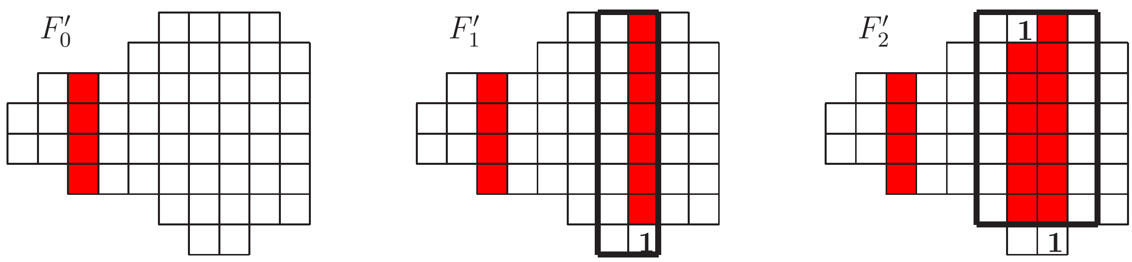
  Then insert m_{i_j} 1's in the i_j-th row of F'_{j-1} in such a way that the number of uncolored cells on this row strictly
  - to the left of the first 1 is $t_{m_{i_j}}$,
  - between the u-th 1 and the $(u+1)$-th 1, $1 \le u \le m_{i_j} - 1$, is $t_{m_{i_j}-u}$,
  - to the right of the last 1 is t_0.

  Next, color the cells which are both contained in the i_j-th rectangle and below (resp., above) the new 1's inserted in R_{i_j} if $R_{i_j} \in Up(T)$ (resp., $R_{i_j} \in Low(T)$).

(3) Set $\Phi(F) = F'_s$. For a better understanding, we give an illustration. Suppose F is the filling given below.



Then the step-by-step construction of $\Phi(F)$ goes as follows.



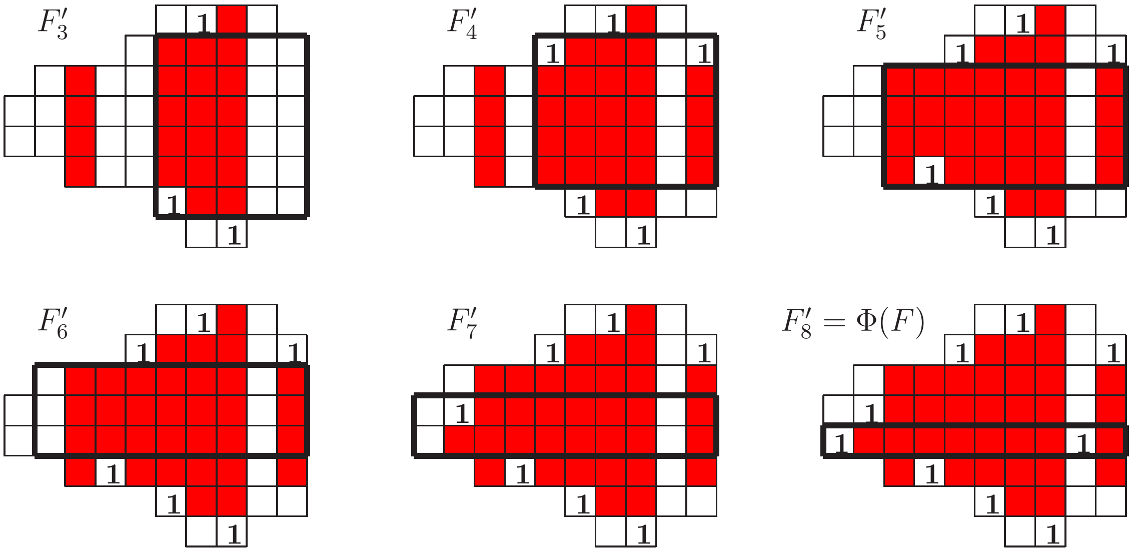


FIGURE 11. The step-by-step construction of $\Phi(F)$

6. Concluding remarks

It is natural, in view of the results obtained in this paper, to ask if the joint distribution of the statistic $(\mathrm{ne}_2, \mathrm{se}_2)$ is symmetric over arbitrary 01-fillings of moon polyominoes, i.e., there are no restrictions on the number of 1's in columns and rows. The answer is no by means of the following result. Given a moon polyomino T, recall that $\mathcal{N}^{01}(T)$ is the set of all 01-fillings of T.

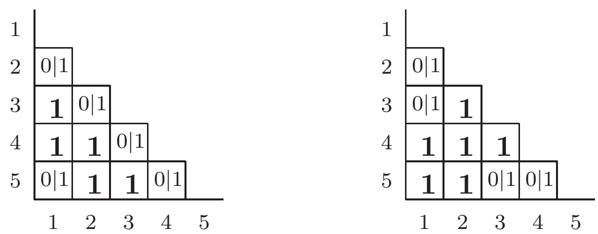
Proposition 6.1. *For any $n \geq 5$ the numbers of arbitrary 01-fillings of Δ_n*

- *with exactly $\binom{n}{4}$ descents is equal to 2^n,*
- *with exactly $\binom{n}{4}$ ascents is equal to 16.*

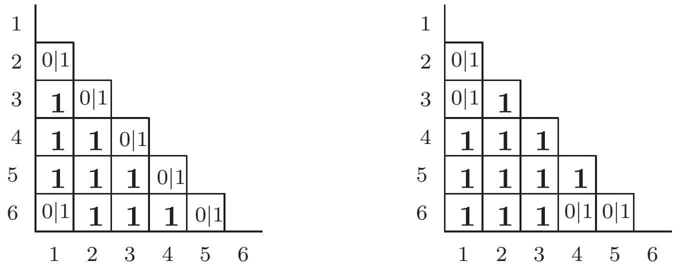
In particular, for any $n \geq 5$, the statistics ne_2 and se_2 are not equidistributed over $\mathcal{N}^{01}(\Delta_n)$, and thus the joint distribution of $(\mathrm{ne}_2, \mathrm{se}_2)$ over $\mathcal{N}^{01}(\Delta_n)$ is not symmetric.

This also implies that the statistics cros_2 and nest_2 are not equidistributed over all simple graphs of $[n]$.

Proof. We give the proof for $n = 5, 6$ since the reasoning can be generalized for arbitrary n. Suppose $n = 5$. Then one can check that the arbitrary 01-fillings of Δ_5 with exactly 5 descents and those with exactly 5 ascents have respectively the following "form"

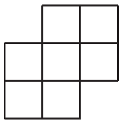


from which it is easy to obtain the result. Similarly, for $n = 6$, the arbitrary 01-fillings of Δ_6 with exactly 15 descents and those with exactly 15 ascents have respectively the following "form"



□

One can also ask if Theorem 2.1, or more generally Corollary 2.4 or Corollary 2.5, can be extended to arbitrary larger classes of polyominoes. We note that the condition of intersection free is necessary. Indeed, the polyomino T represented below is convex but not intersection free,



and

$$\sum_{F \in \mathcal{N}^c(T,(1,1,1))} p^{\mathrm{ne}_2(F)} q^{\mathrm{se}_2(F)} = \sum_{F \in \mathcal{N}(T;3)} p^{\mathrm{ne}_2(F)} q^{\mathrm{se}_2(F)} = p^2 + 2q$$

is not symmetric. One can also check that $\sum_{F \in \mathcal{N}^c(T)} p^{\mathrm{ne}_2(F)} q^{\mathrm{se}_2(F)}$ and $\sum_{F \in \mathcal{N}(T)} p^{\mathrm{ne}_2(F)} q^{\mathrm{se}_2(F)}$ are even not symmetric.

Let T be a moon polyomino and F be a 01-filling a T. Recall that $\mathrm{ne}(F)$ (resp., $\mathrm{se}(F)$) is the largest k for which F has a NE-chain (resp., SE-chain) of length k such that the smallest rectangle containing the chain is contained in F. Rubey [13], answering a conjecture of Jonsson [8], have proved that for any positive integers j and k, we have

$$|\{F \in \mathcal{N}^{01}(T) : |F| = j\,,\ \mathrm{ne}(F) = k\}| = |\{F \in \mathcal{N}^{01}(T^*) : |F| = j\,,\ \mathrm{ne}(F) = k\}| \quad (6.1)$$

for any moon polyomino T^* obtained from T by permutating the columns (or equivalently the rows) of T.

On the other hand, it is easy to derive from Theorem 2.2 the following result.

Proposition 6.2. *Let T be a moon polyomino. For any moon polyomino T^* obtained from T by permutating the rows of T and any positive integers j, k and ℓ, we have*

$$\begin{aligned} &|\{F \in \mathcal{N}^c(T) : |F| = j\,,\ \mathrm{ne}_2(F) = k\,,\ \mathrm{se}_2(F) = \ell\}| \\ =&|\{F \in \mathcal{N}^c(T^*) : |F| = j\,,\ \mathrm{ne}_2(F) = k\,,\ \mathrm{se}_2(F) = \ell\}| \end{aligned}$$

and

$$\begin{aligned} &|\{F \in \mathcal{N}(T) : |F| = j\,,\ \mathrm{ne}_2(F) = k\,,\ \mathrm{se}_2(F) = \ell\}| \\ =&|\{F \in \mathcal{N}(T^*) : |F| = j\,,\ \mathrm{ne}_2(F) = k\,,\ \mathrm{se}_2(F) = \ell\}|. \end{aligned}$$

Clearly the above proposition and Rubey's result (6.1) bring us to the following problem: Is it true that for any moon polyomino T and positive integers j and k we have that

$$|\{F \in \mathcal{N}^{01}(T) : |F| = j\,,\, \mathrm{ne}_2(F) = k\}| = |\{F \in \mathcal{N}^{01}(T^*) : |F| = j\,,\, \mathrm{ne}_2(F) = k\}|$$

for any moon polyomino T^* obtained from T by permutating the rows of T? The answer is no. Indeed, if such a result holds, then by reflecting each moon polyomino in a vertical line and apply the result, we would obtain that the statistics ne_2 and ne_2 are equidistributed over $\mathcal{N}^{01}(T)$ for any moon polyomino T, which contradicts Proposition 6.1.

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